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On Approximation by Semigroups of Nonlinear Contractions, I

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1. INTRODUCTION

The aim of this paper is to study the approximation by a strongly continuous contraction semigroup of nonlinear operators $T(t)$ $(t \geqslant 0)$ of the identity operator I for $t \to 0+$. It is concerned with norm approximation—optimal as well as nonoptimal—in the setting of the theory of interpolation classes constructed by means of the K-functional of J. Peetre [21]. This will be carried out in the framework of an arbitrary Banach space X. The corresponding linear theory is treated fully in P. L. Butzer and H. Berens [6] and H. Berens [2]. For previous nonlinear work in this direction see the note by D. Brézis [4] which is concerned with Hilbert spaces.

In the linear theory the approximation behavior of a semigroup is described by the infinitesimal generator $(-A)$ which is related to it via the differentiability condition

$$Af = s - \lim_{t \to 0+} t^{-1}[f - T(t)f]. \tag{1.1}$$

It is well known that in the nonlinear theory the classical notion of a generator has to be extended (see, e.g., the survey articles of J. R. Dorroh [16] and M. G. Crandall [11] as well as the papers [5], [13–15] and [20]). However, in the setting of an arbitrary Banach space this problem has not as yet been solved in a satisfactory manner. The most general result in this direction, due to M. G. Crandall and T. M. Liggett [13], gives sufficient conditions that an operator A determines a semigroup by the limit

$$T(t)f = s - \lim_{n \to \infty} (I + (t/n)A)^{-n}f.$$

This is an analog of Hille's exponential formula, but does not imply (1.1) in the general nonlinear case. It is under the assumptions of [13] that we will treat the approximation problem in question. This requires that we have to work with the resolvent operator $J_t := (I + tA)^{-1}$ instead of the integral $t^{-1} \int_0^t T(u)\, du$ used in the linear theory. The family of operators J_t $(t \geq 0)$ also defines a strong approximation process towards the identity I, and we shall deduce the approximation assertions for $T(t)$ by comparing both processes. Thereby the estimation of J_t by $T(t)$ depends on an important lemma of Brézis [3]. This paper also contains complete proofs of results announced in [22].

Section 2 is concerned with some notations and the basic results of Crandall and Liggett. In Section 3 we introduce a nonlinear version of the K-functional and compare it with the resolvent and the semigroup operators. While Section 4 gives the results in the intermediate class setting, Section 5 is devoted to relative completion in connection with the saturation problem. An application to the theory, namely to an initial boundary value problem considered by Y. Konishi [18] is left to Note II following the present one.

2. Preliminaries

Let X be a real Banach space, X^* the dual of X, their norms being both denoted by $\|\ \|$, and let (f, f^*) denote the value of $f^* \in X^*$ at $f \in X$. For a nonempty subset $S \subset X$ we set $|S| = \inf\{\|f\|; f \in S\}$.

Let $\mathscr{T} = \{T(t); t \geq 0\}$ be a family of operators from a subset $C \subset X$ into itself satisfying the following conditions

$$T(t + \tau) = T(t)\, T(\tau), \qquad \text{for} \quad t, \tau \geq 0, \qquad T(0) = I, \tag{2.1}$$

$$s - \lim_{t \to 0+} T(t) f = f, \qquad \text{for each} \quad f \in C, \tag{2.2}$$

$$\| T(t) f - T(t) g \| \leq \| f - g \| \qquad \text{for} \quad t \geq 0 \quad \text{and} \quad f, g \in C. \tag{2.3}$$

Then $\mathscr{T}$ is called a contraction semigroup on C, and one writes $\mathscr{T} \in Q(C)$. It follows immediately from (2.1) to (2.3) that $t \to T(t) f$ is a strongly continuous function from $[0, \infty)$ in C for each $f \in C$. Furthermore, one has for an $f \in C$ (cf. [14]): If

$$\liminf_{h \to 0+} h^{-1} \| T(h) f - f \| = L < \infty, \quad \text{then} \quad \| T(t) f - f \| \leq t \cdot L, \qquad (t \geq 0).$$

In the particular case $L = 0$, this implies

$$\| T(t) f - f \| = o(t) \ \ (t \to 0+) \Rightarrow T(t) f = f, \qquad (t \geq 0). \tag{2.4}$$

For purposes of notation let us recall several elementary concepts. A subset $A \subset X \times X$ is called a multivalued operator in X with domain $D(A) = \{f; Af \neq \varnothing\}$ and range $R(A) = \bigcup\{Af; f \in D(A)\}$, where $Af := \{g; [f, g] \in A\}$ for $f \in X$. If B is another multivalued operator in X and λ is real one sets

$$\begin{aligned} A + B &= \{[f, g + h]; [f, g] \in A, [f, h] \in B\}, \\ \lambda A &= \{[f, \lambda g]; [f, g] \in A\}, \\ AB &= \{[f, g]; [f, h] \in B, [h, g] \in A \text{ for some } h \in X\}, \\ A^{-1} &= \{[g, f]; [f, g] \in A\}. \end{aligned}$$

A singlevalued operator A in X is regarded as that special case of a multivalued one for which Af contains exactly one element for each $f \in D(A)$. Let us set $J_\lambda := (I + \lambda A)^{-1}$ and $A_\lambda := \lambda^{-1}(I - J_\lambda)$ for $\lambda \neq 0$; then $D(J_\lambda) = D(A_\lambda) = R(I + \lambda A)$, $R(J_\lambda) = D(A)$. A subset $A \subset X \times X$ is said to be accretive, provided that J_λ is a singlevalued operator for $\lambda > 0$ and

$$\| J_\lambda f - J_\lambda g \| \leqslant \| f - g \| \qquad (f, g \in D(J_\lambda)), \tag{2.5}$$

or in an equivalent form (see Kato [17]), also to be used, A is said to be accretive, if for each $[f_i, g_i] \in A$, $i = 1, 2$, there exists $f^* \in F(f_1 - f_2)$ such that $(g_1 - g_2, f^*) \geqslant 0$. Here F denotes the duality map of X into X^* which is a subset of $X \times X^*$ defined by

$$F(f) = \{f^* \in X^*; (f, f^*) = \| f \|^2 = \| f^* \|^2\},$$

for each $f \in X$.

An accretive operator determines a semigroup in the following sense.

THEOREM 2.1 (Crandall and Liggett [13]). *Let $A \subset X \times X$ be accretive and $R(I + \lambda A) \supset \overline{D(A)}$ for $\lambda > 0$. Then the limit*

$$T(t)f := s - \lim_{n \to \infty} (I + (t/n)A)^{-n} f, \tag{2.6}$$

exists for $f \in \overline{D(A)}$, $t > 0$ and defines a semigroup $\mathscr{T} = \{T(t); t \geqslant 0\} \in Q(\overline{D(A)})$.

If a semigroup $\mathscr{T} \in Q(C)$ is connected with an accretive subset $A \subset X \times X$ by the limit relation (2.6) for each $f \in C$, then one says $\mathscr{T}$ is generated by $(-A)$. We shall also make use of the following facts (see [13, 17]) which are valid under the assumptions of Theorem 2.1,

$$\| T(t)f - f \| \leqslant 2t \, | Af |, \qquad (f \in D(A)), \tag{2.7}$$

$$\| J_\lambda f - f \| = \| \lambda A_\lambda f \| \leqslant \lambda \, | Af |, \qquad (f \in D(A)), \tag{2.8}$$

$$s - \lim_{\lambda \to 0+} J_\lambda f = f, \qquad (f \in \overline{D(A)}), \tag{2.9}$$

$$[J_\lambda f, A_\lambda f] \in A \quad \text{and} \quad | AJ_\lambda f | \leqslant \| A_\lambda f \|, \qquad (f \in \overline{D(A)}). \tag{2.10}$$

3. Basic Comparison Estimates

In view of (2.2) a semigroup $\mathscr{T} \in Q(C)$ is an approximation process tending towards the identity operator I for $t \to 0+$ in the strong topology of X. The aim here is to characterize the approximation behavior of this process by structural properties upon the elements f in X. This will be carried out in case $\mathscr{T}$ is generated by a multivalued operator $(-A)$ in terms of which the structural properties will be expressed. Therefore from now on we assume that the hypotheses of Theorem 2.1 are satisfied. At first we wish to compare the norm of $T(t)f-f$ with that of $J_t f - f$ and to relate the latter to an abstract modulus of continuity given by the K-functional,

$$K(t, f; \overline{D(A)}, D(A)) = K(t, f) := \inf_{g \in D(A)} (\|f-g\| + t\,|Ag|), \qquad (f \in \overline{D(A)},\ t > 0).$$

This is a monotone increasing function with respect to $t \in (0, \infty)$ for each $f \in \overline{D(A)}$, and satisfies,

$$K(t, f) \leqslant t\,|Af|, \qquad (f \in D(A)). \tag{3.1}$$

If A is linear, $K(t, f)$ defines a function seminorm on X (see, e.g., [6, p. 167]). In the linear case it is standard to compare $K(t, f)$ directly with the semigroup operator $T(t)f$, using the fact that $I_t f := t^{-1} \int_0^t T(u) f\, du$ belongs to $D(A)$ and $t^{-1}[T(t)f - f] = AI_t f$ for each $f \in \overline{D(A)}$, A being the infinitesimal generator of the linear semigroup. In the nonlinear situation, however, these properties are in general not valid. We therefore work with the resolvent operator J_t instead of I_t. By (2.10) it has properties corresponding to those of I_t, namely: $J_t f \in D(A)$ and $t^{-1}[f - J_t f] \in AJ_t f$ for each $f \in \overline{D(A)}$. This will allow us to estimate $K(t, f)$ by $\|J_t f - f\|$. To compare $T(t)f$ and $J_t f$ we apply an important result of H. Brézis [3] which was extended by M. G. Crandall and T. M. Liggett [13] and I. Miyadera [19]. Its most general form stated in [19, p. 250, formula (2.11)] reads:

Lemma 3.1. *Let the hypotheses of Theorem* 2.1 *be satisfied. If* $f \in \overline{D(A)}$ *and* $[f_0, g_0] \in A$, *then*

$$(T(t)f - f, \xi^*) \leqslant \int_0^t \langle g_0, f_0 - T(\tau)f \rangle_s \, d\tau, \tag{3.2}$$

for each $t \geqslant 0$ *and each* $\xi^* \in F(f - f_0)$. *Here*

$$\langle g_0, f_0 - T(\tau)f \rangle_s := \sup\{(g_0, f^*); f^* \in F(f_0 - T(\tau)f)\} = (g_0, f_\tau^*),$$

where $f_\tau{}^*$ is an element of $F(f_0 - T(\tau)f)$ for which the supremum is actually attained.

Whereas [13] and [19] used formula (3.2) for $t \to 0+$ to deduce

$$\sup_{\xi^* \in F(f-f_0)} \limsup_{t\to 0+} \left(\frac{T(t)f - f}{t}, \xi^*\right) \leqslant \langle g_0, f_0 - f\rangle_s,$$

we will avoid taking limits since we need (3.2) for each fixed $t > 0$ in order to establish the inequality (3.5) below. Moreover, an elementary inequality (see [16]) will be used in the following:

Inequality. For any $a, b \in X, f^* \in F(a)$ and $g^* \in F(b)$ one has

$$2(a - b, g^*) \leqslant \| a \|^2 - \| b \|^2 \leqslant 2(a - b, f^*). \tag{3.3}$$

THEOREM 3.2. *Under the assumptions of Theorem* 2.1 *one has for $f \in \overline{D(A)}$ and all $t > 0$*

$$\| T(t)f - f \| \leqslant 4 \| J_t f - f \|, \tag{3.4}$$

$$\| J_t f - f \| \leqslant 2t^{-1} \int_0^t \| T(\tau)f - f \| \, d\tau + 2 \| T(t)f - f \|, \tag{3.5}$$

$$K(t, f) \leqslant 2 \| J_t f - f \|, \tag{3.6}$$

$$\| J_t f - f \| \leqslant 2K(t, f). \tag{3.7}$$

Proof. By (2.3), (2.7) and (2.10) one has

$$\begin{aligned} \| T(t)f - f \| &\leqslant \| T(t)f - T(t) J_t f \| + \| T(t) J_t f - J_t f \| + \| J_t f - f \| \\ &\leqslant 2 \| J_t f - f \| + 2t \, | AJ_t f | \leqslant 4 \| J_t f - f \|, \end{aligned}$$

yielding (3.4). For the proof of (3.5) we make use of (3.2) with $f_0 = J_t f$ and $g_0 = A_t f$. Then one has

$$(T(t)f - f, \xi^*) \leqslant t^{-1} \int_0^t (f - J_t f, f_\tau{}^*) \, d\tau, \tag{3.8}$$

for each $\xi^* \in F(f - J_t f)$ and some $f_\tau{}^* \in F(J_t f - T(\tau) f)$, $t > 0$ being fixed. The left-hand side of (3.8) may be estimated from below by Schwarz' inequality by

$$(T(t)f - f, \xi^*) \geqslant -\| T(t)f - f \| \, \| J_t f - f \|, \tag{3.9}$$

noting that $\| \xi^* \| = \| J_t f - f \|$. For the right-hand side of (3.8) we use the first inequality in (3.3) with $a = f - T(\tau)f$, $b = J_t f - T(\tau)f$ and $g^* = f_\tau^*$. This yields

$$2(f - J_t f, f_\tau^*) \leqslant \| T(\tau)f - f\|^2 - \| J_t f - T(\tau)f\|^2.$$

Moreover, since

$$\| J_t f - T(\tau)f\|^2 \geqslant \| J_t f - f\|^2 + \| T(\tau)f - f\|^2 - 2\| J_t f - f\| \, \| T(\tau)f - f\|,$$

one has

$$2(f - J_t f, f_\tau^*) \leqslant [-\| J_t f - f\| + 2\| T(\tau)f - f\|] \, \| J_t f - f\|. \quad (3.10)$$

Combining the estimates from above and below for (3.8), namely (3.9) and (3.10), one obtains (3.5) by dividing the resulting inequality by $\| J_t f - f\|$. Inequality (3.6) follows from,

$$K(t, f) \leqslant \| f - g\| + t \, | Ag |, \qquad (g \in D(A)),$$

if g is taken as $J_t f$, noting (2.10). Concerning (3.7), let $g \in D(A)$ be arbitrary. Then $g = J_t(g + tg')$ for each $g' \in Ag$, and by (2.5)

$$\begin{aligned} \| J_t f - f\| &\leqslant \| J_t f - J_t(g + tg')\| + \| f - g\| \\ &\leqslant \| f - g - tg'\| + \| f - g\| \leqslant 2(\| f - g\| + t\| g'\|). \end{aligned}$$

Taking the infimum with respect to all $g' \in Ag$ followed by that with respect to all $g \in D(A)$, (3.7) now holds, and the proof is complete.

As a first application of Theorem 3.2 we obtain a characterization of those elements $f \in \overline{D(A)}$ which are approximated by $T(t)f$ with order $O(t^\alpha)$.

COROLLARY 3.3. *Under the hypotheses of Theorem* 2.1 *the following are equivalent for an* $f \in \overline{D(A)}$:

(i) $\| T(t)f - f\| = O(t^\alpha)$,

(ii) $\| J_t f - f\| = O(t^\alpha)$,

(iii) $K(t, f) = O(t^\alpha)$.

Note that these assertions are only of interest in case $0 < \alpha \leqslant 1$, just as in the linear situation. Indeed, if $\alpha > 1$, then (i) implies that $T(t)f$ approximates f with order $o(t)$, $t \to 0+$, giving $T(t)f = f$ for each $t \geqslant 0$ by (2.4). Further characterizations of (i) that are only valid in case $\alpha = 1$ are left to Section 5. We also refer to the concluding remarks there concerning an interpretation of the two different cases $0 < \alpha < 1$ and $\alpha = 1$, as well as for the history to the matter.

4. Intermediate Classes

The purpose here is to formulate the above results in the framework of intermediate classes. To this end, we now gather all elements having a definite approximation behavior into a set. First, let us introduce the functional $\Phi_{\alpha,q}$ ($\alpha > 0$, $1 \leqslant q \leqslant \infty$) defined on the set of positive measurable functions $g = g(t)$, $0 < t < 1$,

$$\Phi_{\alpha,q}(g(t)) := \begin{cases} \left[\int_0^1 (t^{-\alpha} g(t))^q \, dt/t\right]^{1/q}, & (1 \leqslant q < \infty), \\ \operatorname*{ess\,sup}_{0<t<1} (t^{-\alpha} g(t)), & (q = \infty). \end{cases}$$

Definition 4.1. For $\alpha > 0$, $1 \leqslant q \leqslant \infty$ we define

$$[A]^T_{\alpha,q} := \{f \in \overline{D(A)};\ \Phi_{\alpha,q}(\| T(t)f - f \|) < \infty\},$$
$$[A]^J_{\alpha,q} := \{f \in \overline{D(A)};\ \Phi_{\alpha,q}(\| J_t f - f \|) < \infty\},$$
$$[A]^K_{\alpha,q} := \{f \in \overline{D(A)};\ \Phi_{\alpha,q}(K(t, f)) < \infty\}.$$

Before showing the connection between these classes let us mention some of their elementary properties which, for simplicity, we only state for the sets $[A]^T_{\alpha,q}$.

Lemma 4.2.

(a) $[A]^T_{\alpha_1,q} \subset [A]^T_{\alpha_2,q}$ $(0 < \alpha_2 < \alpha_1)$,

(b) $[A]^T_{\alpha,q_1} \subset [A]^T_{\alpha,q_2}$ $(1 \leqslant q_1 \leqslant q_2 \leqslant \infty)$,

(c) *If* $f \in [A]^T_{\alpha,q}$, *then, as* $t \to 0+$,

$$\| T(t)f - f \| = \begin{cases} o(t^\alpha) & (1 \leqslant q < \infty), \\ O(t^\alpha) & (q = \infty). \end{cases}$$

(d) *For* $0 < \alpha < 1$, $1 \leqslant q < \infty$ *and* $0 < \alpha \leqslant 1$, $q = \infty$ *one has*

$$D(A) \subset [A]^T_{\alpha,q} \subset \overline{D(A)}. \tag{4.1}$$

In particular, for $f \in D(A)$,

$$\Phi_{\alpha,q}(\| T(t)f - f \|) \leqslant \begin{Bmatrix} ((1-\alpha)q)^{-1/q} \\ 1 \end{Bmatrix} 2 \, | Af | \qquad \begin{matrix} (1 \leqslant q < \infty), \\ (q = \infty). \end{matrix}$$

The proofs follow along the standard lines of the linear theory. Note that it is only necessary to study the sets $[A]^T_{\alpha,q}$ for those values of α and q which are specified in part (d) (cf. the remarks following Corollary 3.3). Furthermore observe that (4.1) states that the classes $[A]^T_{\alpha,q}$ are intermediate between $D(A)$ and $\overline{D(A)}$. We now come to one main result of the paper.

THEOREM 4.3. *Let $A \subset X \times X$ be accretive and $\overline{D(A)} \subset R(I + \lambda A)$ for $\lambda > 0$. If $\mathscr{T} = \{T(t);\, t \geqslant 0\}$ is the semigroup generated by $(-A)$, then for $0 < \alpha < 1$, $1 \leqslant q < \infty$ and $0 < \alpha \leqslant 1$, $q = \infty$,*

$$[A]^T_{\alpha,q} = [A]^J_{\alpha,q} = [A]^K_{\alpha,q}.$$

In particular,

$$\Phi_{\alpha,q}(\| T(t)f - f\|) \leqslant 4\Phi_{\alpha,q}(\| J_t f - f\|) \leqslant 8\Phi_{\alpha,q}(K(t,f)) \tag{4.2}$$

and

$$\begin{aligned} \Phi_{\alpha,q}(K(t,f)) &\leqslant 2\Phi_{\alpha,q}(\| J_t f - f\|) \\ &\leqslant \begin{Bmatrix} [(\alpha q + 1)^{-1/q} + 1] \\ [(\alpha + 1)^{-1} + 1] \end{Bmatrix} 4\Phi_{\alpha,q}(\| T(t)f - f\|) \quad \begin{matrix} (1 \leqslant q < \infty), \\ (q = \infty). \end{matrix} \end{aligned} \tag{4.3}$$

The proof follows from Theorem 3.2. Concerning the second part of (4.3), which results from (3.5), note that by Hölder's inequality for $\alpha > 0$

$$\begin{aligned} &\Phi_{\alpha,q}\left(t^{-1}\int_0^t \| T(\tau)f - f\|\, d\tau\right) \\ &\qquad \leqslant \begin{Bmatrix} (\alpha q + 1)^{-1/q} \\ (\alpha + 1)^{-1} \end{Bmatrix} \Phi_{\alpha,q}(\| T(t)f - f\|) \qquad \begin{matrix} (1 \leqslant q < \infty), \\ (q = \infty). \end{matrix} \end{aligned}$$

Observe that the case $q = \infty$, $0 < \alpha \leqslant 1$ is already covered by Corollary 3.3. Theorem 4.3 seems to be the first result on nonlinear semigroup approximation in the setting of intermediate classes contained in a not necessarily reflexive Banach space. In the case of a Hilbert space H we refer to a note of D. Brézis [4]; particularly compare his inequalities

$$\tfrac{1}{3}\| T(t)f - f\|_H \leqslant \| J_t f - f\|_H \leqslant 3\| T(t)f - f\|_H,$$

with (3.4) and (3.5).

5. RELATIVE COMPLETION AND SATURATION

For $\alpha = 1$, $q = \infty$ there is a further characterization of $[A]^T_{\alpha,q}$, namely via the concept of relative completion, the linear version of which was introduced by E. Gagliardo (see [1]). In the framework of approximation theory it was first used by H. Berens [2] (see also [8]).

DEFINITION 5.1. The completion of $D(A)$ relative to $\overline{D(A)}$, denoted by $\widetilde{D(A)}$, is the set of elements $f \in \overline{D(A)}$ for which there exists a sequence $\{f_n\} \subset D(A)$ such that

$$s - \lim_{n\to\infty} f_n = f, \tag{5.1}$$

$$| Af_n | \leqslant M \text{ for all } n, M \text{ being independent of } n. \tag{5.2}$$

Before proceeding we need some further notions concerning accretive operators A: If $D(A) \subset S \subset X$, A is called maximal accretive on S if A is accretive and any accretive extension of A coincides on S with A, i.e., if $B \subset X \times X$ is accretive and $A \subset B$, then $Af = Bf$ for each $f \in S$. A is said to be m-accretive (hyper-accretive) if A is accretive and $R(I + \lambda_0 A) = X$ for some $\lambda_0 > 0$. A is called almost demiclosed if $[f_n, g_n] \in A$ $(n = 1, 2,...)$, $s - \lim_{n\to\infty} f_n = f$, $w - \lim_{n\to\infty} g_n = g$ imply $f \in D(A)$. If, in addition, $g \in Af$, A is called demiclosed.

For the next proposition we use a counterpart of Lemma 3.1 for the resolvent operator J_λ, the proof being simple.

LEMMA 5.2. *Let A be accretive. If $f \in D(J_\lambda)$ and $[f_0, g_0] \in A$, then for each $\lambda > 0$ and each $\xi^* \in F(f_0 - f)$*

$$(f - J_\lambda f, \xi^*) \leqslant \lambda \langle g_0, f_0 - J_\lambda f \rangle_s . \tag{5.3}$$

Proof. Since $[f_0, g_0] \in A$ and $[J_\lambda f, \lambda^{-1}(f - J_\lambda f)] \in A$, A being accretive, there exists an $\eta^* \in F(f_0 - J_\lambda f)$ such that

$$(f - J_\lambda f, \eta^*) \leqslant \lambda(g_0, \eta^*).$$

The right-hand side of this inequality may be estimated from above by $\lambda\langle g_0, f_0 - J_\lambda f\rangle_s$, the left-hand side from below by $(f - J_\lambda f, \xi^*)$ for each $\xi^* \in F(f_0 - f)$, the latter following, e.g., from (3.3) if one substitutes there $a = f_0 - J_\lambda f$, $b = f_0 - f$, $f^* = \eta^*$ and $g^* = \xi^*$. Thus (5.3) is proven.

In reflexive spaces X the relative completion may be characterized by Proposition 5.3.

PROPOSITION 5.3. *Let X be reflexive and let the hypothesis of Theorem 2.1 be satisfied.*

(a) *If A is almost demiclosed, then $\widetilde{D(A)} = D(A)$.*

(b) *If A is demiclosed or maximal accretive on $\overline{D(A)}$, then*

$$| Af | = \lim_{\lambda\to 0+} \| A_\lambda f \| = \lim_{\lambda\to 0+} \lambda^{-1} \| J_\lambda f - f \| \qquad (f \in D(A)). \tag{5.4}$$

Proof. That $D(A) \subset \widetilde{D(A)}$ is obvious. If $f \in \widetilde{D(A)}$, then there exists $\{f_n\} \subset D(A)$ satisfying (5.1) and (5.2). For each $\lambda > 0$ one has

$$\| A_\lambda f \| = \lim_{n\to\infty} \| A_\lambda f_n \| \leqslant \limsup_{n\to\infty} \| Af_n \| \leqslant M, \tag{5.5}$$

by (5.1), (2.8) and (5.2). Since X is reflexive, each sequence $\{A_{\lambda_m} f\}$, $\lim_{m\to\infty} \lambda_m = 0$, contains a subsequence $\{A_{\lambda_{m_i}} f\}$ such that

$$w - \lim_{i\to\infty} A_{\lambda_{m_i}} f = g, \tag{5.6}$$

for some $g \in X$. Now, if A is almost demiclosed, then $f \in D(A)$, proving (a). This part would also follow by applying Lemma 3.8 in [17]. Concerning (b), if in addition, A is demiclosed, then $g \in Af$ and

$$\| Af \| \leqslant \| g \| \leqslant \liminf_{i\to\infty} \| A_{\lambda_{m_i}} f \| \leqslant \limsup_{i\to\infty} \| A_{\lambda_{m_i}} f \| \leqslant \| Af \|.$$

This yields

$$\lim_{i\to\infty} \| A_{\lambda_{m_i}} f \| = \| Af \| = \| g \|,$$

for any sequence $\{\lambda_m\}$, and thus (5.4) follows. The same conclusion is valid if A is maximal accretive on $\overline{D(A)}$, provided one can show that $g \in Af$. For this purpose we make use of Lemma 5.2. Thus for each $[f_0, g_0] \in A$ and each $\xi^* \in F(f_0 - f)$ one has by (5.6)

$$(g, \xi^*) \leqslant \limsup_{i\to\infty} \langle g_0, f_0 - J_{\lambda_{m_i}} f \rangle_s \leqslant \langle g_0, f_0 - f \rangle_s,$$

the latter inequality following since the map $\langle \cdot, \cdot \rangle_s \colon X \times X \to R$ is upper semicontinuous (cf. [13]). Now, there exists $h^* \in F(f_0 - f)$ such that $\langle g_0, f_0 - f \rangle_s = (g_0, h^*)$, $F(f_0 - f)$ being weak* compact. Hence

$$(g, \xi^*) \leqslant (g_0, h^*).$$

Since A is maximal accretive, one may apply Lemma 3.4 in [17] to the latter inequality, giving that $[f, g] \in A$. This completes the proof of the proposition.

The following theorem gives the connection between the notion of relative completion and the intermediate sets of the foregoing section.

THEOREM 5.4. *Under the same assumptions as in Theorem* 4.3 *one has* $[A]^T_{1,\infty} = \widetilde{D(A)}$.

Proof. If $f \in [A]^T_{1,\infty}$, then $\sup_{0<t<1} t^{-1} \| T(t)f - f \| = M_0 < \infty$. By (3.5) it follows that

$$\| A_t f \| \leqslant 2t^{-2} \int_0^t \tau M_0 \, d\tau + 2M_0 = 3M_0 .$$

$\{J_t f\}$ is a family of elements from $D(A)$ satisfying (5.1) and (5.2) by (2.9) and (2.10) and therefore $f \in \widetilde{D(A)}$. If, conversely, $f \in \widetilde{D(A)}$, then by the same arguments which yielded (5.5) we have $\| A_t f \| \leqslant M$ for all $t > 0$, M being a constant. This gives by (3.4)

$$\Phi_{1,\infty}(\| T(t)f - f \|) \leqslant 4M,$$

implying $f \in [A]^T_{1,\infty}$.

Theorem 5.4 (and Theorem 4.3 for $\alpha = 1$) combined with the $o(t)$-assertion in (2.4) gives a result on saturation—or on the so-called optimal approximation—of the process $\mathscr{T} = \{T(t); t \geqslant 0\}$ for $t \to 0+$.

COROLLARY 5.5.

(a) *Under the assumptions of Theorem* 2.1 *the semigroup* $\mathscr{T} \in Q(\overline{D(A)})$ *generated by* $(-A)$ *is saturated with order* $O(t)$, *and its saturation (Favard) class* $[A]^T_{1,\infty}$ *is characterized equivalently not only by* $[A]^K_{1,\infty}$ *and* $[A]^J_{1,\infty}$, *respectively, but also by* $\widetilde{D(A)}$.

(b) *If, in addition, X is reflexive and A maximal accretive on* $\overline{D(A)}$, *then* $[A]^T_{1,\infty} = D(A)$.

Thus, interpreted in the framework of approximation theory, Theorem 4.3 yields for $\alpha = 1$ an equivalence theorem on optimal approximation and for values α, $0 < \alpha < 1$, an equivalence theorem on nonoptimal approximation. Corollary 5.5 in the setting of a nonreflexive Banach space X was announced in the author's note [22] answering a question posed by P. L. Butzer and J. R. Dorroh on the occasion of an Oberwolfach Conference (cf. [7]). In this connection, Crandall [12] showed that

$$\liminf_{t \to 0+} t^{-1} \| T(t)f - f \| = \lim_{\lambda \to 0+} \lambda^{-1} \| J_\lambda f - f \|, \tag{5.7}$$

for each $f \in \overline{D(A)}$, and regarded the set of those f for which (5.7) is finite as a "generalized domain" of A. This result was forwarded to the author after the appearance of [22]. For the linear background to the Crandall result see P. L. Butzer and S. Pawelke [9]. Part (b) of Corollary 5.5 for the reflexive case is to be found in Miyadera [19]. Previously Brézis [3] had proved this result under the additional assumptions that A is m-accretive and X^* uniformly convex.

However, the investigations of these authors and others in this field (see, e.g., [10]) were not so much concerned with the approximation theoretical point of view but with the differentiability of the semigroup generated by $(-A)$, still an open problem in the general nonreflexive case.

Perhaps the viewpoint of optimal approximation of this paper may be of help in these investigations.

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